Solvable multispecies reaction-diffusion processes

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A family of one-dimensional multispecies reaction-diffusion processes on a lattice is introduced. It is shown that these processes are exactly solvable, provided a nonspectral matrix equation is satisfied. Some general remarks on the solutions to this equation, and some special solutions are given. The large-time behavior of the conditional probabilities of such systems is also investigated.

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I. INTRODUCTION

In recent years, the asymmetric exclusion process and the problems related to it, including, for example, bipolymerization [1], dynamical models of interface growth [2], traffic models [3], the noisy Burgers equation [4], and the study of shocks [5,6], have been extensively investigated. The dynamical properties of this model have been studied in [6–8]. As the results obtained by approaches like mean field are not reliable in one dimension, it is useful to introduce solvable models and analytic methods to extract exact physical results. Among these methods is the coordinate Bethe ansatz, which was used in [9] to solve the asymmetric simple exclusion process on a one-dimensional lattice. In [10], a similar technique was used to solve the drop-push model [11] and a generalized one-parameter model interpolating between the asymmetric simple exclusion model and the drop-push model. In [12], this family was further generalized to a family of processes with arbitrary left- and right-diffusion rates. All of these models were lattice models. Finally, the behavior of the last model on a continuum was investigated in [13]. Continuum models of this kind were also investigated in [14,15]. In [16] a generalization of such processes was studied that also contained annihilation of particles.

In all of these studies, people have been mainly concerned with so-called single-species processes, in which only one kind of particle exists and moves on the lattice (or the continuum). Another interesting problem is the study of multispecies systems where several kinds of particle move and interact on a lattice. In [17], single-species systems were characterized; the equations governing the evolution of the N-point functions contain N- or fewer-point functions. This was done for multispecies systems in [18]. In [19], two-species reaction-diffusion systems were introduced that are solvable in the sense that the S matrix corresponding to them is factorizable into two-particle S matrices. It was found there that the criterion for this is that the interactions (which are of the nearest-neighbor type) must be such so that the S matrix satisfies a kind of spectral Yang-Baxter equation.

We follow the same line. That is, we investigate interactions that can be written as boundary conditions for the probability functions. These interactions preserve the total number of particles, so that if one begins with N particles, knowing the N-point probabilities is enough to know everything about the system. Using the coordinate Bethe ansatz, it is found that for this ansatz to be consistent the S matrix should satisfy a kind of spectral Yang-Baxter equation [19]. However, the S matrix is of a special form containing the boundary conditions (or interactions), and not every solution of the spectral Yang-Baxter equation can be used to construct such a solvable model. We investigate the spectral equation the S matrix should satisfy and show that this is equivalent to a nonspectral equation for the boundary conditions. This is independent of the number of species.

The scheme of the paper is the following. In Sec. II, the prescription for investigating multispecies reaction-diffusion systems in terms of diffusion systems equipped with suitable boundary conditions is studied. In Sec. III, the Bethe-ansatz solution for such (solvable) systems is obtained and its large-time behavior is investigated. In Sec. IV, the solvability criterion is obtained and it is shown that this criterion is a nonspectral matrix equation. In Sec. V, some general properties of the solutions of the solvability criterion are studied. Finally, in Sec. VI some special solutions of the solvability equation are studied.

II. MULTISPECIES REACTION-DIFFUSION SYSTEMS AND THE BOUNDARY CONDITIONS

Consider a system consisting of N particles on a lattice, drifting to the right with unit rate if the right neighboring site is empty, and interacting with each other only if two of them are adjacent. Suppose that there are n kinds (or species) of particle and the interaction between the particles is in the form that, if two particles A^α and A^β are adjacent to each other, they may change to A^γ and A^δ with the rate $b_{\alpha\beta}^{\gamma\delta}$. That is, the allowed processes are

$$A^\alpha\varnothing\rightarrow\varnothing A^\alpha \quad \text{with rate } 1$$

$$A^\alpha A^\beta\rightarrow A^\gamma A^\delta \quad \text{with rate } b_{\alpha\beta}^{\gamma\delta}, \tag{1}$$

where

$$\sum_{\gamma\delta} b_{\alpha\beta}^{\gamma\delta}=1. \tag{2}$$

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These processes result in the master equation

$$\dot{P}^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N;t) = P^{\alpha_1,\ldots,\alpha_N}(x_1-1,\ldots,x_N;t)+\cdots + P^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N-1;t) - NP^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N;t), \quad (3)$$

if $x_i<x_{i+1}-1$. The symbol $P^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N;t)$ denotes the probability of finding a particle of type α_1 in x_1, a particle of type α_2 in $x_2,\ldots$ at the time t. The so-called physical region consists of the points satisfying $x_i<x_{i+1}$. If $x_i=x_{i+1}-1$, the interactions change the equation. For clarity, let us write the evolution equation for the two-particle sector:

$$\dot{P}^{\alpha\beta}(x,x+1)=P^{\alpha\beta}(x-1,x+1) + \sum_{(\gamma\delta)\neq(\alpha\beta)} b^{\alpha\beta}_{\gamma\delta}P^{\gamma\delta}(x,x+1) - B^{\alpha\beta}P^{\alpha\beta}(x,x+1)-P^{\alpha\beta}(x,x+1), \quad (4)$$

where

$$B^{\alpha\beta}:=\sum_{(\gamma\delta)\neq(\alpha\beta)} b^{\gamma\delta}_{\alpha\beta}. \quad (5)$$

Using

$$b^{\alpha\beta}_{\alpha\beta}:=1-\sum_{(\gamma\delta)\neq(\alpha\beta)} b^{\gamma\delta}_{\alpha\beta}, \quad (6)$$

it is seen that Eq. (4) can be written as

$$\dot{P}^{\alpha\beta}(x,x+1)=P^{\alpha\beta}(x-1,x+1)+b^{\alpha\beta}_{\gamma\delta}P^{\gamma\delta}(x,x+1) - 2P^{\alpha\beta}(x,x+1), \quad (7)$$

where summation is implied on repeated indices. Comparing this with Eq. (3), it is seen that it can be written as Eq. (3) provided one introduces the boundary condition

$$P^{\alpha\beta}(x,x)=b^{\alpha\beta}_{\gamma\delta}P^{\gamma\delta}(x,x+1), \quad (8)$$

or, in a more compact form,

$$|P(x,x)\rangle=b\ \ |P(x,x+1)\rangle. \quad (9)$$

The matrix b should satisfy two criteria. First, its nondiagonal elements should be non-negative (since they are rates). Second, the sum of the elements of each of its columns should be 1. This can be written in a compact form as

$$\langle s|\otimes\langle s|b=\langle s|\otimes\langle s|, \quad (10)$$

where

$$s_\alpha:=1. \quad (11)$$

Note that if the number of species is 1 the asymmetric simple exclusion process [9] is obtained.

III. THE BETHE-ANSATZ SOLUTION

As in [19], one can write a Bethe-ansatz solution for Eq. (3) with the boundary condition

$$|P(\ldots,x_k=x,x_{k+1}=x,\ldots)\rangle = b_{k,k+1}|P(\ldots,x_k=x,x_{k+1}=x+1,\ldots)\rangle, \quad (12)$$

where

$$b_{k,k+1}:=1\otimes\cdots\otimes 1\otimes\underbrace{b}_{k,k+1}\otimes 1\otimes\cdots\otimes 1. \quad (13)$$

We take the ansatz

$$|P(\mathbf{x};t)\rangle=e^{Et}|\Psi(\mathbf{x})\rangle, \quad (14)$$

and it is seen that $|\Psi(\mathbf{x})\rangle$ should satisfy

$$E|\Psi(x_1,\ldots,x_N)\rangle=|\Psi(x_1-1,\ldots,x_N)\rangle+\cdots + |\Psi(x_1,\ldots,x_N-1)\rangle - N|\Psi(x_1,\ldots,x_N)\rangle \quad (15)$$

and

$$|\Psi(\ldots,x_k=x,x_{k+1}=x,\ldots)\rangle = b_{k,k+1}|\Psi(\ldots,x_k=x,x_{k+1}=x+1,\ldots)\rangle. \quad (16)$$

The Bethe ansatz is that one takes the following form for $|\Psi(\mathbf{x})\rangle$:

$$|\Psi(\mathbf{x})\rangle=\sum_\sigma A_\sigma e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}|\psi\rangle, \quad (17)$$

where $|\psi\rangle$ is an arbitrary vector and the summation runs over the elements of the permutation group. Putting this in Eq. (15) results in

$$E=\sum_{k=1}^{N}(e^{-ip_k}-1). \quad (18)$$

The boundary condition (16) yields

$$[1-e^{i\sigma(p_{k+1})}b_{k,k+1}]A_\sigma+[1-e^{i\sigma(p_k)}b_{k,k+1}]A_{\sigma\sigma_k}=0. \quad (19)$$

Here σ_k is that element of the permutation group which only interchanges p_k and p_{k+1}. From this, one obtains

$$A_{\sigma\sigma_k}=S_{k,k+1}[\sigma(p_k),\sigma(p_{k+1})]A_\sigma, \quad (20)$$

where the matrix S is defined through

$$S(p_1,p_2):=-(1-z_1b)^{-1}(1-z_2b), \quad (21)$$

and the definition of $S_{k,k+1}$ is similar to that of $b_{k,k+1}$ in Eq. (13). We have also used the definition

$$z_j:=e^{ip_j}. \quad (22)$$

This shows that one can construct A_σ's from A_1 by writing σ as a product of σ_k's. But these elements of the permutation group satisfy

$$\sigma_k\sigma_{k+1}\sigma_k=\sigma_{k+1}\sigma_k\sigma_{k+1}. \tag{23}$$

This means that

$$A_{\sigma_k\sigma_{k+1}\sigma_k}=A_{\sigma_{k+1}\sigma_k\sigma_{k+1}}, \tag{24}$$

or

$$\begin{aligned}&S_{k,k+1}(p_{k+1},p_{k+2})S_{k+1,k+2}(p_k,p_{k+2})S_{k,k+1}(p_k,p_{k+1})\\&\quad=S_{k+1,k+2}(p_k,p_{k+1})S_{k,k+1}(p_k,p_{k+2})\\&\qquad\times S_{k+1,k+2}(p_{k+1},p_{k+2}).\end{aligned} \tag{25}$$

This can be written as

$$\begin{aligned}&[S(p_2,p_3)\otimes 1][1\otimes S(p_1,p_3)][S(p_1,p_2)\otimes 1]\\&\quad=[1\otimes S(p_1,p_2)][S(p_1,p_3)\otimes 1][1\otimes S(p_2,p_3)].\end{aligned} \tag{26}$$

Writing the S matrix as the the product of the permutation matrix Π and an R matrix,

$$S_{k,k+1}=:\Pi_{k,k+1}R_{k,k+1}, \tag{27}$$

Equation (26) is transformed to

$$\begin{aligned}&R_{23}(p_2,p_3)R_{13}(p_1,p_3)R_{12}(p_1,p_2)\\&\quad=R_{12}(p_1,p_2)R_{13}(p_1,p_3)R_{23}(p_2,p_3).\end{aligned} \tag{28}$$

This is the spectral Yang-Baxter equation.

Provided this condition is satisfied, it is easy to see that the conditional probability (the propagator) is

$$U(\mathbf{x};t|\mathbf{y};0)=\int\frac{d^Np}{(2\pi)^N}\,e^{-i\mathbf{p}\cdot\mathbf{y}}\sum_\sigma A_\sigma e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}e^{tE(\mathbf{p})}, \tag{29}$$

where the integration region for each p_i is from $[0,2\pi]$, and we have taken $A_e=1$. (e is the identity of the permutation group.) Note that Eq. (10) and the condition of nonnegativity of the nondiagonal elements of b ensure that the absolute values of the eigenvalues of b other than 1 are not equal to 1. So there is no singularity in $S(p_1,p_2)$ except at $p_1=0$, and this is removed by setting $p_j\to p_j+i\epsilon$, where one should consider the limit $\epsilon\to 0^+$. This is the same as was done in [9] and [10], for example. Using this propagator, one can of course write the probability at the time t in terms of the initial value of the probability:

$$|P(\mathbf{x};t)\rangle=\sum_{\mathbf{y}}U(\mathbf{x};t|\mathbf{y};0)|P(\mathbf{y};0)\rangle. \tag{30}$$

For the two-particle sector, it is not difficult to obtain U. In fact, as there is only one matrix (b) in the expression for U, one can treat it as a c number and the problem is reduced to that of [16], with λ replaced by b. So,

$$\begin{aligned}U(\mathbf{x};t|\mathbf{y};0)=&e^{-2t}\frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}\\&+e^{-2t}\sum_{l=0}^{\infty}\frac{t^{l+x_2-y_1}}{(l+x_2-y_1)!}\frac{t^{x_1-y_2}}{(x_1-y_2)!}\\&\times b^l\left(-1+\frac{tb}{x_1-y_2+1}\right).\end{aligned} \tag{31}$$

One can decompose the vector space on which b acts into a subspace on which $b=1$ (eigenspace of b corresponding to eigenvalue 1) and another invariant subspace. This is done by decomposing the unit matrix into two projectors

$$1=Q+R, \tag{32}$$

where Q and R are projections satisfying

$$QR=RQ=0. \tag{33}$$

Q is the projection on the eigenspace of b corresponding to the eigenvalue 1, and R is the projection on the other invariant subspace of b. Using this, one can write U as

$$\begin{aligned}U(\mathbf{x};t|\mathbf{y};0)=&\left[e^{-2t}\frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}+e^{-2t}\sum_{l=0}^{\infty}\frac{t^{l+x_2-y_1}}{(l+x_2-y_1)!}\frac{t^{x_1-y_2}}{(x_1-y_2)!}\left(-1+\frac{t}{x_1-y_2+1}\right)\right]Q\\&+\left[e^{-2t}\frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}+e^{-2t}\sum_{l=0}^{\infty}\frac{t^{l+x_2-y_1}}{(l+x_2-y_1)!}\frac{t^{x_1-y_2}}{(x_1-y_2)!}b^l\left(-1+\frac{tb}{x_1-y_2+1}\right)\right]R.\end{aligned} \tag{34}$$

Here we have used

$$b=b(Q+R)=Q+bR. \tag{35}$$

As the eigenvalues of b, other than 1, are assumed to have moduli not equal to 1, the second term in Eq. (34) is the same as Eq. (33) in [16], that is, a term obtained from the boundary condition corresponding to annihilation ($\lambda<1$ in [16]). The first term corresponds to an asymmetric simple exclusion process [9]. The large-time behavior of these two terms is also simply obtained. The large-time behavior of the first was obtained in [13], and that of the second in [16]. At large times, the second term is found to be independent of b

(or λ) and vanishing faster than $1/t$. Also, the summation of this term vanishes as t tends to infinity. In fact, using [16] it is seen that

$$(\text{the second term}) = \frac{1}{2\pi t}\{e^{-[(x_1-y_1-t)^2+(x_2-y_2-t)^2]/(2t)} - e^{-[(x_1-y_1-t)^2+(x_2-y_2-t)^2]/(2t)}\}, \quad t\to\infty. \tag{36}$$

So at large times only the first term of Eq. (34) survives. This means that at large times the propagator is proportional to the projection on the eigenspace of b corresponding to the eigenvalue 1 (the projection on the *equilibrium* subspace of b) and the proportionality constant is simply the propagator of the asymmetric simple exclusion process.

To conclude, for large times the two-particle conditional probability is that of an asymmetric simple exclusion process projected on the eigenspace of b corresponding to its unit eigenvalue.

IV. SOLVABILITY CRITERIA FOR THE BOUNDARY CONDITIONS

From Eq. (21), it is seen that $S(p_1,p_2)$ is a binomial of degree 1 with respect to $z_2 := e^{ip_2}$. Putting this in Eq. (26), one arrives at a quadratic expression with respect to z_3. The coefficients of this expression are, of course, matrices depending on z_1 and z_2. It is easy to find the roots of this expression for z_3. In fact, putting $z_3 = z_1$ in Eq. (26), one arrives at the identity

$$[S(p_2,p_1)\otimes 1][S(p_1,p_2)\otimes 1] \equiv [1\otimes S(p_1,p_2)][1\otimes S(p_2,p_1)]. \tag{37}$$

[We note that $S(p_1,p_2)S(p_2,p_1)\equiv 1$.] Also, putting $z_3 = z_2$, another identity is obtained:

$$[1\otimes S(p_1,p_2)][S(p_1,p_2)\otimes 1] \equiv [1\otimes S(p_1,p_2)][S(p_1,p_2)\otimes 1]. \tag{38}$$

These two identities show that the roots of the quadratic expression for z_3 are z_1 and z_2. That is, one can write that expression as

$$(z_3-z_1)(z_3-z_2)Q(z_1,z_2)=0. \tag{39}$$

So Eq. (26) is equivalent to $Q=0$, which itself is obtained by putting $z_3=0$ in Eq. (26):

$$\begin{aligned}&[(1-z_2b)^{-1}\otimes 1][1\otimes(1-z_1b)^{-1}]\\ &\times[(1-z_1b)^{-1}(1-z_2b)\otimes 1]\\ &=[1\otimes(1-z_1b)^{-1}(1-z_2b)][(1-z_1b)^{-1}\otimes 1]\\ &\times[1\otimes(1-z_2b)^{-1}].\end{aligned} \tag{40}$$

Inverting both sides, one arrives at

$$\begin{aligned}&[(1-z_2b)^{-1}(1-z_1b)\otimes 1][1\otimes(1-z_1b)][(1-z_2b)\otimes 1]\\ &=[1\otimes(1-z_2b)][(1-z_1b)\otimes 1]\\ &\times[1\otimes(1-z_2b)^{-1}(1-z_1b)].\end{aligned} \tag{41}$$

This is a quadratic expression in terms of z_1. For $z_1=0$, Eq. (41) gives the identity

$$[(1-z_2b)^{-1}\otimes 1][(1-z_2b)\otimes 1] \equiv [1\otimes(1-z_2b)][1\otimes(1-z_2b)^{-1}], \tag{42}$$

while for $z_1=z_2$ the identity

$$[1\otimes(1-z_2b)][(1-z_2b)\otimes 1] \equiv [1\otimes(1-z_2b)][(1-z_2b)\otimes 1] \tag{43}$$

is obtained. So the quadratic expression corresponding to Eq. (41) is equivalent to

$$z_1(z_1-z_2)\tilde{Q}(z_2)=0, \tag{44}$$

and to find $\tilde{Q}$ one simply uses the coefficient of z_1^2 in Eq. (41). This is

$$(1-z_2b_{12})^{-1}b_{12}b_{23}(1-z_2b_{12}) = (1-z_2b_{23})b_{12}b_{23}(1-z_2b_{23})^{-1}, \tag{45}$$

or

$$b_{12}b_{23}(1-z_2b_{12})(1-z_2b_{23})=(1-z_2b_{12})(1-z_2b_{23})b_{12}b_{23}. \tag{46}$$

This is a quadratic expression in z_2. But the coefficients of z_2^0 and z_2^2 are identities. So the only remaining equation is

$$b_{12}b_{23}(b_{12}+b_{23})=(b_{12}+b_{23})b_{12}b_{23}, \tag{47}$$

or

$$b_{12}[b_{12},b_{23}]=[b_{12},b_{23}]b_{23}. \tag{48}$$

Equation (48) is equivalent to Eq. (26). But it is seen that Eq. (48) is nonspectral, whereas Eq. (26) is spectral. That is, Eq. (26) is an equation to be satisfied for all values of certain parameters, whereas there are no parameters in Eq. (48). So it is far simpler to seek the solutions to Eq. (48) than to seek those of Eq. (26).

To summarize, a matrix b, or the reactions (1), corresponds to an exactly solvable reaction-diffusion system on a one-dimensional lattice, provided b satisfies Eqs. (48) and (6) [or Eq. (10), equivalently], and the nondiagonal elements of b are non-negative.

V. GENERAL PROPERTIES OF THE SOLUTIONS TO THE SOLVABILITY CRITERIA

Solutions to Eq. (48) have two general properties. First, if b is a solution, then

$$b' := \alpha b + \beta \tag{49}$$

is another solution for constant α and β. If b satisfies Eq. (10), then

$$\langle s|\otimes\langle s|b'=(\alpha+\beta)\langle s|\otimes\langle s|. \tag{50}$$

So putting $\beta:=1-\alpha$ ensures that b' satisfies Eq. (10). If $\alpha>0$, then the nondiagonal elements of b' are non-negative provided the nondiagonal elements of b are non-negative. So

$$b':=\alpha b+(1-\alpha) \tag{51}$$

corresponds to a solvable system (for $\alpha>0$) if b does. It is easy to see that the meaning of this transformation is simply to multiply the reaction rates by α.

Second, if b is a solution to Eq. (48), then

$$b':=u\otimes ubu^{-1}\otimes u^{-1} \tag{52}$$

satisfies Eq. (48) as well. Here u is an arbitrary (nonsingular) matrix. This transformation, however, does not necessarily respect the conditions (10) and non-negativity of the rates. Thus another problem arises. Suppose b is a solution to Eq. (48), and we want to obtain a solvable system using the transformation (52). We must have

$$\langle s|\otimes\langle s|u\otimes ub=\langle s|\otimes\langle s|u\otimes u. \tag{53}$$

This means that u must change $\langle s|$ to some $\langle s'|$ so that $\langle s'|\otimes\langle s'|$ is a left eigenvector of b corresponding to a unit eigenvalue. One may search in the eigenvectors of b to find whether there is an eigenvector of the form $\langle s'|\otimes\langle s'|$. If there is such an eigenvector, then any matrix u that changes $\langle s|$ to $\langle s'|$ can be used to obtain b' according to Eq. (52). This b' satisfies Eqs. (48) and (10). But its diagonal elements may be non-negative or not; this should be checked separately. If none of the eigenvalues of b are of the form $\langle s'|\otimes\langle s'|$, then this method cannot be used to obtain a solvable system. This method resembles very closely that used in [20].

VI. SOME SPECIAL CASES

Case I: $b^2=\alpha+\beta b$ (α and β are numbers). In this case, one can define

$$b':=b+\gamma, \tag{54}$$

with γ satisfying

$$\gamma^2+\beta\gamma-\alpha=0, \tag{55}$$

to obtain

$$b'^2=(\beta+2\gamma)b'. \tag{56}$$

Putting this b' in Eq. (48), one obtains the braid equation for b':

$$b'_{12}b'_{23}b'_{12}=b'_{23}b'_{12}b'_{23}. \tag{57}$$

From Eq. (56), it is seen that b' either can be scaled to a projection ($b'^2=b'$), or is nilpotent. One concludes then that any nilpotent or projection solution to the (nonspectral) braid equation is a solution to Eq. (48). One can then use any linear combination of this solution with the unit matrix as another solution to that equation. Note, however, that these solutions of Eq. (48) do not necessarily satisfy other criteria of the solvable system, that is, non-negativity of the nondiagonal elements and Eq. (10). An inspection of the solutions obtained in [19] shows that solutions (1)–(15) and (17) are of this type. As mentioned in the previous section, one can of course take a linear combination of each solution with the unit matrix to obtain another solution.

Case II: $b=u\otimes v$. Here Eq. (48) takes the form

$$u^2\otimes v[v,u]\otimes v=u\otimes[v,u]u\otimes v^2. \tag{58}$$

A simple way to satisfy this is to set

$$[u,v]=0. \tag{59}$$

Thus, using any two commuting matrices u and v one can construct a solution to Eq. (48). If the elements of one of these matrices are non-negative, and the nondiagonal elements of the other are also non-negative, then the nondiagonal elements of b are non-negative. If

$$\langle s|u=\langle s|v=\langle s|, \tag{60}$$

then b satisfies Eq. (10) as well. Of course, having found a solution of this type one can use a linear combination of it with the unit matrix as another solution. Solutions (1), (4), (7), (14), (17), (20), (21), (22), (25), (26), and (28) of [19] are of this kind.

It is possible to have other solutions to Eq. (58). In this case, let us also use Eq. (10). This shows that one may rescale u and v so that Eq. (60) is satisfied. One then arrives at

$$u^2=u,$$

$$v^2=v,$$

$$vuv=uvu, \tag{61}$$

if $[u,v]\neq 0$. From these, it is seen that u, v, $1-u$, and uvu are projections. Moreover,

$$(1-u)uvu=uvu(1-u)=0. \tag{62}$$

This shows that $1-u$ and uvu can be simultaneously diagonalized. The diagonal form of them will be

$$1-u=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$uvu=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}. \tag{63}$$

Here the elements of the above matrices are matrices themselves, and 1 is the unit matrix of the appropriate dimension. Writing an ansatz for v,

$$v=\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}, \tag{64}$$

and putting it in Eq. (61), one finally arrives at the following forms for u and v:

$$u=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$v=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & w' & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{65}$$

where all of the entries in the above matrices are matrices, and w and w' should satisfy

$$ww'=w'w=0. \tag{66}$$

Each of the diagonal blocks of these matrices may be zero dimensional, except the first. It should, at least, be one dimensional. The reason is that u and v have at least one common left eigenvector $\langle s|$ corresponding to the unit eigenvalue. Also, the dimension of each block of u is equal to that of the corresponding block in v. Also note that if the dimension of u and v is 2 (there are two kinds of particle) then there will be no space left for w and w', and u and v must be commuting.

The final result is that in two dimensions no new solution exists (u and v must be commuting), and in more than two dimensions u and v must be of the form of Eq. (65). Of course any similarity transformation on Eq. (65) gives another solution to Eq. (48). In fact, one has to use a similarity transformation to make $\langle s|$ a left eigenvector of u and v with unit eigenvalue.

Two very simple subcases are $b=1\otimes v$ and $b=u\otimes 1$. These describe reactions

$$A^\alpha A^\beta \rightarrow A^\alpha A^\delta \quad \text{with rate } v_\beta^\delta \tag{67}$$

and

$$A^\alpha A^\beta \rightarrow A^\gamma A^\beta \quad \text{with rate } u_\alpha^\gamma, \tag{68}$$

respectively. That is, in each case only one of the particles changes, and the rate of change is independent of the type of the other particle.

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